Quantum integrability of the deformed elliptic Calogero–Moser problem

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Abstract: The integrability of the deformed quantum elliptic Calogero-Moser problem introduced by Chalykh, Feigin and Veselov is proven. Explicit recursive formulae for the integrals are found. For integer values of the parameter this implies the algebraic integrability of the systems.

Key words: Quantum integrability, deformed Calogero–Moser system.

1 Introduction

Deformed quantum Calogero–Moser (CM) systems were introduced by Chalykh, Feigin and Veselov [1, 2], who proved their integrability in rational and trigonometric cases and conjectured that the same is true in the elliptic case. The aim of this paper is to prove this conjecture.

Elliptic deformed CM system corresponds to the following Schrödinger operator

$$L_m^{(n)} = \sum_{j=1}^n \frac{\hat{p}_j^2}{m_j} + 2m(m+1) \sum_{j<k}^n m_j m_k \wp\left(x_j - x_k\right), \tag{1}$$

where all but one "masses" are equal, $m_1 = m^{-1}$, $m_2 = \ldots = m_n = 1$, m is a real parameter, $\hat{p}_j = i\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, and $\wp$ is the classical Weierstrass elliptic function. The case when m is integer is a special one: in that case a stronger version of integrability (the so-called algebraic integrability) was conjectured [2]. The first results in this direction were found in [3], where it was proven in the simplest non-trivial case $n = 3, m = 2$.

The main result of the present paper is an explicit recursive formula for the quantum integrals of the elliptic deformed CM system. This proves integrability of the system for all n and m and due to a general recent result by Chalykh, Etingof and Oblomkov [4] this also implies the algebraic integrability for integer values of the parameter m.

As a by-product we have also new formulae for the integrals of the usual quantum elliptic CM problem, which was the subject of many investigations since 1970s (see in particular [5], [6], [7, 8], [9, 10, 11]). We will be using some technical tricks from these important papers. In trigonometric and rational limits we have the formulae for the quantum integrals of the corresponding deformed CM systems which are also seem to be new.

2 Preliminaries and main formulae

Quantum Hamiltonian of the deformed elliptic CM problem has the following form

$$H = -\left(m\partial_1^2 + \partial_2^2 + \ldots + \partial_{n-1}^2 + \partial_n^2\right) + 2\left(m+1\right)\sum_{k=2}^{n}\wp_{1k} + 2m\left(m+1\right)\sum_{2\leq j<k\leq n}\wp_{jk}, \tag{2}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $\wp_{jk} = \wp\left(x_j - x_k\right)$. Here $\wp$ is the classical Weierstrass elliptic function [12], which can be determined by the differential equation

$$\left(\wp'\left(z\right)\right)^2 = 4\left(\wp\left(z\right) - e_1\right)\left(\wp\left(z\right) - e_2\right)\left(\wp\left(z\right) - e_3\right) = 4\wp^3\left(z\right) - g_2\wp\left(z\right) - g_3. \tag{3}$$

The Laurent expansion of $\wp$ at the origin is of the following form [12]

$$\wp\left(z\right) = z^{-2} + \sum_{k=1}^{\infty}\gamma_{2k}z^{2k}, \tag{4}$$

where

$$\gamma_2 = \frac{1}{20}g_2, \quad \gamma_4 = \frac{1}{28}g_3.$$

The coefficients γ_{2k} are related to the so-called *Bernoulli-Hurwitz numbers* $BH\left(k\right)$ [13]

$$\gamma_{2k} = \frac{1}{(2k)!}\frac{BH\left(2k+2\right)}{(2k+2)}.$$

There is a recursive formula which allows one to obtain the coefficients γ_{2k+2} from the coefficients of the lower order:

$$\gamma_{2k+2} = \frac{3}{(k-1)(2k+5)}\sum_{j=1}^{k-1}\gamma_{2j}\gamma_{2k-2j}, \quad k = 2, 3, \ldots. \tag{5}$$

The relationship (5) is easy to verify. One needs to differentiate (3) to obtain

$$\wp''\left(z\right) = 6\wp^2\left(z\right) - \frac{1}{2}g_2,$$

use the expansion (4) and collect terms at the appropriate degrees of z.

To construct the integrals of the operator (2) we will follow the idea going back to [6], [7]. Namely, the integrals are constructed from the highest one by successive commutators with some function which in our case is $m^{-1}x_1 + x_2 + \ldots + x_n$. This highest integral is of order n and will be denoted below as I. The rest of this section is to explain the main ingredients of the formula for I.

Let us introduce the following differential operators $\mathfrak{D}^k$ in ∂_1 with constant coefficients

$$\mathfrak{D}^1 = \partial_1, \quad \mathfrak{D}^2 = \frac{(1-m)}{2!}\partial_1^2, \quad \mathfrak{D}^3 = \frac{(1-m)(1-2m)}{3!}\partial_1^3, \quad \mathfrak{D}^4 = \frac{(1-m)(1-2m)(1-3m)}{4!}\partial_1^4,$$

$$\mathfrak{D}^k = p_{0,k}\partial_1^k + \sum_{i=2}^{\left[\frac{k}{2}\right]}p_{2i,k}\partial_1^{k-2i}. \tag{6}$$

The constants $p_{0,k}$, $k = 2, 3, \ldots$, are given by

$$p_{0,k} = \tfrac{1}{k!} \prod_{l=1}^{k-1} (1 - lm), \quad k = 2, 3, \ldots, \tag{7}$$

and constants $p_{2i,k}$, $i = 2, 3, \ldots$, $k = 2i, 2i + 1, \ldots$, are determined by the following recursive relationship

$$p_{2i,2i} = 0, \quad i = 2, 3, \ldots, \tag{8}$$

$$p_{2i,k} = \tfrac{(1 - m(k - 2i - 1))}{k - 2i} p_{2i,k-1} - (1 + m) \sum_{\substack{j=1 \\ j \neq 2}}^{i-1} (2i - 2j)! C_{k-2j+1}^{k-2i} \gamma_{2i-2j} p_{2j-2,k-1}, \tag{9}$$

$$i = 2, 3, \ldots, \quad k = 2i + 1, 2i + 2, \ldots$$

and $C_n^k = \frac{n!}{k!(n-k)!}$ is a binomial coefficient. To illustrate the formula (9) let us write explicitly the formulae for the first few values of i:

$$p_{4,k} = \tfrac{(1 - m(k - 5))}{k - 4} p_{4,k-1} - (1 + m) \tfrac{\prod_{l=1}^3 (k-l)}{3} \gamma_2 p_{0,k-1}, \quad k = 5, 6, \ldots,$$

$$p_{6,k} = \tfrac{(1 - m(k - 7))}{k - 6} p_{6,k-1} - (1 + m) \tfrac{\prod_{l=1}^5 (k-l)}{5} \gamma_4 p_{0,k-1}, \quad k = 7, 8, \ldots,$$

$$p_{8,k} = \tfrac{(1 - m(k - 9))}{k - 8} p_{8,k-1} - (1 + m) \left(\tfrac{\prod_{l=1}^7 (k-l)}{7} \gamma_6 p_{0,k-1} - \tfrac{\prod_{l=5}^7}{3} \gamma_2 p_{4,k-1} \right), \quad k = 9, 10, \ldots.$$

It is interesting to note that for the special values of parameter $m = \tfrac{1}{l}$, where l is a positive integer number, most of the constants (7) and (9) are zero. For example, if $m = \tfrac{1}{2}$ only $p_{0,2}$ is non zero, if $m = \tfrac{1}{3}$ then only $p_{0,2}$ and $p_{0,3}$ are non zero, if $m = \tfrac{1}{4}$ then only $p_{0,2}$, $p_{0,3}$, $p_{0,4}$ and $p_{4,5}$, $p_{4,6} = p_{0,2}p_{4,5}$, $p_{4,7} = p_{0,3}p_{4,5}$, $p_{4,8} = p_{0,4}p_{4,5}$ are non zero and so on.

Let us introduce the following notations:

$$\varsigma_j = (m + 1) \varsigma (x_1 - x_j), \quad 2 \leq j \leq n,$$
$$u_{1j} = (m + 1) \wp (x_1 - x_j), \quad 2 \leq j \leq n,$$
$$u_{kl} = m (m + 1) \wp (x_k - x_l), \quad 2 \leq k < l \leq n,$$

where ς is the standard elliptic ς-function: $\frac{d\varsigma(z)}{dz} = -\wp(z)$.

We will need also to consider all the subsystems of the deformed CM system. Let S be a subset of the set $\{1, 2, \ldots, n\}$ and $\sigma = \{j_1, j_2, \ldots, j_t\}$, $j_1 < j_2 < \ldots < j_t$, be a subset of t indices chosen from S. The set of all different subsets σ of size t of the set S will be denoted by

$$\mathfrak{S}(S; t) = \{\sigma = \{j_1, j_2, \ldots, j_t\} : j_1 < j_2 < \ldots < j_t, \ j_k \in S\}.$$

If $\sigma \in \mathfrak{S}(S; t)$ define the set $S \backslash \sigma = \{j : j \in S \text{ and } j \notin \sigma\}$. If $S = \{1, 2, \ldots, n\}$ we will use short notation $\mathfrak{S}(t) = \mathfrak{S}(\{1, 2, \ldots, n\}; t)$ and $\hat{\sigma} = \{1, 2, \ldots, n\} \backslash \sigma$. If σ contains only one element $\sigma = \{k\}$ the brackets will be omitted: k will denote a set which contains one element $\{k\}$, $S \backslash k = S \backslash \{k\}$ and $\hat{k} = \{1, 2, \ldots, n\} \backslash \{k\}$. We will also use notation $\hat{\sigma}_1 \hat{\sigma}_2$ to denote the intersection of the subsets $\hat{\sigma}_1$ and $\hat{\sigma}_2$: $\hat{\sigma}_1 \hat{\sigma}_2 = \hat{\sigma}_1 \cap \hat{\sigma}_2$.

We use the notation $ad^t_{\varsigma_\sigma}\left(\mathfrak{D}^k\right)$ to denote the repeated commutator

$$ad^t_{\varsigma_\sigma}\left(\mathfrak{D}^k\right) = \left[\varsigma_{i_t}, \dots \left[\varsigma_{i_2}, \left[\varsigma_{i_1}, \mathfrak{D}^k\right]\right]\right], \qquad \sigma = \{j_1, j_2, \dots, j_t\}. \tag{10}$$

Note that the order in which ς_{i_k} are used is not important because of the form of the operator $\mathfrak{D}^k$. Define

$$\Theta = \mathfrak{D}^n + \sum_{t=1}^{\left[\frac{n}{2}\right]} \sum_{\sigma \in \mathfrak{S}\left(\hat{1};t\right)} ad^t_{\varsigma_\sigma}\left(\mathfrak{D}^{n-t}\right) = \sum_{t=0}^{\left[\frac{n}{2}\right]} \sum_{\sigma \in \mathfrak{S}\left(\hat{1};t\right)} ad^t_{\varsigma_\sigma}\left(\mathfrak{D}^{n-t}\right).$$

Let the set S consist of k elements and contain 1, then define the operator Θ_S as

$$\Theta_S = \mathfrak{D}^k + \sum_{t=1}^{\left[\frac{k}{2}\right]} \sum_{\sigma \in \mathfrak{S}(S\backslash 1;t)} ad^t_{\varsigma_\sigma}\left(\mathfrak{D}^{k-t}\right) = \sum_{t=0}^{\left[\frac{k}{2}\right]} \sum_{\sigma \in \mathfrak{S}(S\backslash 1;t)} ad^t_{\varsigma_\sigma}\left(\mathfrak{D}^{k-t}\right). \tag{11}$$

We also use the notation ∂_S to denote the product $\partial_S = \prod_{j \in S} \partial_j$. If $1 \in S$ then Δ_S denotes the sum $\Delta_S = m\partial_1^2 + \sum_{j \in S\backslash 1} \partial_j^2$. If $1 \notin S$ then $\Delta_S = \sum_{j \in S} \partial_j^2$. By I_S we will mean the corresponding quantum integral of the system with the Hamiltonian

$$H_S = -\Delta_S + 2 \sum_{\substack{j<k \\ j,k \in S}} u_{jk}.$$

Now we are ready to give the formula for the integrals:

$$I = \sum_{t=1}^{n-2} (-1)^{t+1} \sum_{\sigma \in \mathfrak{S}(t)} I_{\hat{\sigma}}\partial_\sigma + (-1)^n (n-1) \partial_1 \dots \partial_n + X, \tag{12}$$

where

$$X = \Theta + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}\left(\hat{1};2t\right)} X_\sigma \Theta_{\hat{\sigma}} \tag{13}$$

and X_σ are related to a non deformed CM subsystem and are determined by the recurrent formulae

$$X_{\hat{1}} = \sum_{j=2}^{n-1} X_{\{j,n\}}X_{\hat{1}\backslash\{j,n\}}, \text{ if } n = 2p-1; \ X_{\hat{1}} = 0, \text{ if } n = 2p \text{ and } X_{\{2,3\}} = u_{23}. \tag{14}$$

Theorem 1. *The operator I defined by (12) and (13) commutes with the deformed elliptic CM operator H.*

Remark. The formula (12) is valid in the non deformed case also: the operator

$$I_{\hat{1}} = \sum_{t=1}^{n-3} (-1)^{t+1} \sum_{\sigma \in \mathfrak{S}\left(\hat{1},t\right)} I_{\hat{\sigma}}\partial_\sigma + (-1)^n (n-2) \partial_2 \dots \partial_n + X_{\hat{1}}$$

commutes with the operator

$$H_{\hat{1}} = -\left(\partial_2^2 + \dots + \partial_{n-1}^2 + \partial_n^2\right) + 2m(m+1) \sum_{2 \le j < k \le n} \wp_{jk},$$

which is the usual $n-1$ particle elliptic CM operator.

Idea of the proof of Theorem 1. The proof will be done by induction. The main idea behind formula (12) consists in the observation that one can use the commutativity of $I_{\hat{\sigma}}\partial_{\sigma}$ with $H_{\hat{\sigma}}$ to simplify the commutator $[I, H]$ to the expression

$$[I, H] = [X, H] + \sum_{j=1}^{n} \left[X_{\hat{j}}\partial_{j}, 2\sum_{\substack{l=1\\l\neq j}}^{n} u_{jl} \right] + \sum_{1\leq k < l \leq n} \left[X_{\hat{k}\hat{l}}\partial_{k}\partial_{l}, 2u_{kl} \right],$$

where terms X, $X_{\hat{j}}$ and $X_{\hat{k}\hat{l}}$ depend only on ∂_1 and u_{sr}, $1 \leq s < r \leq n$. This is shown in Lemma 1 in section 6. At the next step we notice that if X is given by (13) the commutator $[I, H]$ can be simplified further to the following expression

$$[I, H] = [\Theta, H] + \sum_{j=2}^{n} \left[\Theta_{\hat{j}}\partial_{j}, 2\sum_{\substack{l=1\\l\neq j}}^{n} u_{jl} \right] + \sum_{2\leq k < l \leq n} \left[\Theta_{\hat{k}\hat{l}}, 2\left(u_{1k} + u_{1l}\right) \right] X_{\{k,l\}}$$

where Θ, $\Theta_{\hat{j}}$ and $\Theta_{\hat{k}\hat{l}}$ depend only on ∂_1 and u_{1s}, $2 \leq s \leq n$. From this we can deduce that

$$\frac{\partial \Theta}{\partial x_k} = \left[u_{1k}, \Theta_{\hat{k}} \right]$$

and, therefore, it seems natural to use the operators $ad_{\varsigma_{\sigma}}^{t}$ to construct Θ. At this stage the only freedom left is in choosing operators $\mathfrak{D}^k$ which must be the operators in ∂_1 with constant coefficients. To ensure that $[I, H] = 0$ these operators must satisfy the relation

$$[\mathfrak{D}^n, \wp\left(x_1 - x_i\right)] + (1 + m)\left[\left[\varsigma\left(x_1 - x_i\right), \mathfrak{D}^{n-1}\right], \wp\left(x_1 - x_i\right)\right]$$
$$+ m\left[\mathfrak{D}^{n-1}, \wp\left(x_1 - x_i\right)\right]\partial_1 - \frac{(1+m)}{2}\wp'\left(x_1 - x_i\right)\mathfrak{D}^{n-1} - \frac{(1-m)}{2}\mathfrak{D}^{n-1}\wp'\left(x_1 - x_i\right) = 0.$$

which is equivalent to a large set of identities. It is remarkable that the constants in $\mathfrak{D}^k$ can be chosen in such a way that all these identities is satisfied (Lemma 3 of Section 6). The choice of the constants is related to the Bernoulli-Hurwitz numbers and is described above. The complete proof of the theorem is quite technical and is given in a separate section.

3 Examples: formulae for two, three and four particles.

To illuminate our formulae let us consider more explicitly the case of small n.

3.1 Two-particle case.

In that case we have the operator

$$H = -m\partial_1^2 - \partial_2^2 + 2\left(m + 1\right)\wp_{12} = -m\partial_1^2 - \partial_2^2 + 2u_{12}$$

which is trivially integrable since the operator $\partial_1 + \partial_2$ obviously commutes with it. This system gives the formula for operator $I_{\{1,2\}}$ which starts the recursive construction of the integral I (12). We have

$$I = I_{\{1,2\}} = \tfrac{1}{2}\left(H + \left(\partial_1 + \partial_2\right)^2\right) = \partial_1\partial_2 + \tfrac{(1-m)}{2}\partial_1^2 + \left(m + 1\right)\wp_{12} = \partial_1\partial_2 + \mathfrak{D}^2 + u_{12} = \partial_1\partial_2 + \Theta,$$

$$X = X_{\{1,2\}} = \Theta_{\{1,2\}} = \mathfrak{D}^2 + u_{12} = \mathfrak{D}^2 + [\varsigma_2, \mathfrak{D}].$$

3.2 Three-particle case.

The integrals for the problem of three particles have been found in [14]. The operator of the third order has the form

$$
\begin{aligned}
I &= \partial_1\partial_2\partial_3 + \frac{(1-m)}{2}(\partial_2 + \partial_3)\partial_1^2 + \frac{(1-m)(1-2m)}{3!}\partial_1^3 + (m+1)\left(m\wp_{23}\partial_1 + \wp_{13}\partial_2 + \wp_{12}\partial_3\right) \\
&\quad + \frac{(1-m)}{2}(m+1)\left((\wp_{12} + \wp_{13})\,\partial_1 + \partial_1\,(\wp_{12} + \wp_{13})\right) \\
&= \partial_1\partial_2\partial_3 + p_{0,2}(\partial_2 + \partial_3)\partial_1^2 + u_{23}\partial_1 + u_{13}\partial_2 + u_{12}\partial_3 + \mathfrak{D}^3 + \left[\varsigma_2, \mathfrak{D}^2\right] + \left[\varsigma_3, \mathfrak{D}^2\right],
\end{aligned}
$$

Operator I can be rewritten as

$$
I = I_{\{2,3\}}\partial_1 + I_{\{1,3\}}\partial_2 + I_{\{1,2\}}\partial_3 - 2\partial_1\partial_2\partial_3 + X,
$$

where

$$
X = \Theta = \mathfrak{D}^3 + \left[\varsigma_2, \mathfrak{D}^2\right] + \left[\varsigma_3, \mathfrak{D}^2\right], \text{ and } I_{\{2,3\}} = \partial_2\partial_3 + u_{23}.
$$

3.3 Four-particle case.

One can check by direct calculation that the operator

$$
\begin{aligned}
I &= \partial_1\partial_2\partial_3\partial_4 + \frac{(1-m)}{2}\left(\partial_2\partial_3 + \partial_2\partial_4 + \partial_3\partial_4\right)\partial_1^2 \\
&\quad + \frac{(1-m)(1-2m)}{3!}\left(\partial_2 + \partial_3 + \partial_4\right)\partial_1^3 + \frac{(1-m)(1-2m)(1-3m)}{4!}\partial_1^4 \\
&\quad + m(m+1)\left(\wp_{34}\partial_1\partial_2 + \wp_{24}\partial_1\partial_3 + \wp_{23}\partial_1\partial_4\right) + (m+1)\left(\wp_{14}\partial_2\partial_3 + \wp_{13}\partial_2\partial_4 + \wp_{12}\partial_3\partial_4\right) \\
&\quad + \frac{(1-m)}{2}(m+1)\partial_1\left((\wp_{13} + \wp_{14})\,\partial_2 + (\wp_{12} + \wp_{14})\,\partial_3 + (\wp_{12} + \wp_{13})\,\partial_4\right) \\
&\quad + \frac{(1-m)}{2}(m+1)\left((\wp_{13} + \wp_{14})\,\partial_2 + (\wp_{12} + \wp_{14})\,\partial_3 + (\wp_{12} + \wp_{13})\,\partial_4\right)\partial_1 \\
&\quad + \frac{(1-m)(1-2m)}{3!}(m+1)\left((\wp_{12} + \wp_{13} + \wp_{14})\,\partial_1^2 + \partial_1\,(\wp_{12} + \wp_{13} + \wp_{14})\,\partial_1 + \partial_1^2\,(\wp_{12} + \wp_{13} + \wp_{14})\right) \\
&\quad + \frac{(1-m)}{2}m(m+1)\left(\wp_{34} + \wp_{24} + \wp_{23}\right)\partial_1^2 \\
&\quad + (1-m)\,(m+1)^2\left(\wp_{14}\wp_{24} + \wp_{14}\wp_{34} + \wp_{24}\wp_{34}\right) + m\,(m+1)^2\left(\wp_{12}\wp_{34} + \wp_{13}\wp_{24} + \wp_{14}\wp_{23}\right)
\end{aligned}
$$

commutes with H. The operator I can be rewritten in the following recursive form

$$
\begin{aligned}
I &= 3\partial_1\partial_2\partial_3\partial_4 + X + I_{\{2,3,4\}}\partial_1 + I_{\{1,3,4\}}\partial_2 + I_{\{1,2,4\}}\partial_3 + I_{\{1,2,3\}}\partial_4 \tag{15} \\
&\quad - I_{\{3,4\}}\partial_1\partial_2 - I_{\{2,4\}}\partial_1\partial_3 - I_{\{2,3\}}\partial_1\partial_4 - I_{\{1,4\}}\partial_2\partial_3 - I_{\{1,3\}}\partial_2\partial_4 - I_{\{1,2\}}\partial_3\partial_4,
\end{aligned}
$$

where

$$
\begin{aligned}
X &= \Theta + X_{\{2,3\}}\Theta_{\{1,4\}} + X_{\{2,4\}}\Theta_{\{1,3\}} + X_{\{3,4\}}\Theta_{\{1,2\}} \\
&= \mathfrak{D}^4 + \left[\varsigma_2, \mathfrak{D}^3\right] + \left[\varsigma_3, \mathfrak{D}^3\right] + \left[\varsigma_4, \mathfrak{D}^3\right] + \left[\varsigma_2, \left[\varsigma_3, \mathfrak{D}^2\right]\right] + \left[\varsigma_2, \left[\varsigma_4, \mathfrak{D}^2\right]\right] + \left[\varsigma_3, \left[\varsigma_4, \mathfrak{D}^2\right]\right] \\
&\quad + u_{23}\left(\mathfrak{D}^2 + \left[\varsigma_4, \mathfrak{D}\right]\right) + u_{24}\left(\mathfrak{D}^2 + \left[\varsigma_3, \mathfrak{D}\right]\right) + u_{34}\left(\mathfrak{D}^2 + \left[\varsigma_2, \mathfrak{D}\right]\right).
\end{aligned}
$$

and

$$I_{\{2,3,4\}} = \partial_1 \partial_2 \partial_3 + u_{23}\partial_1 + u_{13}\partial_2 + u_{12}\partial_3.$$

4 Integrability of the deformed elliptic quantum CM problem.

Let us introduce the function $\theta = m^{-1}x_1 + x_2 + \ldots + x_n$ and consider the corresponding "ad"-operation:

$$ad_\theta(L) = [\theta, L].$$

Following to the procedure known for the usual CM system (see, for example, [8]) consider the operators $L_k = ad_\theta^k(I)$, $k = 0, 1, \ldots, n-1$.

Theorem 2. *Operators $L_k = ad_\theta^k(I)$, $k = 0, 1, \ldots, n-1$, where I are given by (12) commute with each other and with operator H.*

Proof. The proof is similar to the non-deformed case [8, 11].

Let us first prove that L_k commute with H and with $\sum_{j=1}^n \partial_j$. Proof is by induction in k. For $k = 0$ it follows from the Theorem 1. Let us assume that this is true for $k = i$, then for $k = i + 1$ we have by the Jacobi identity

$$[H, L_{i+1}] = [H, [\theta, L_i]] = [[H, \theta], L_i] + [[L_i, H], \theta] = 2\left[\sum_{j=1}^n \partial_j, L_i\right] + [0, \theta] = 0$$

and

$$\left[\sum_{j=1}^n \partial_j, L_{i+1}\right] = \left[\sum_{j=1}^n \partial_j, [\theta, L_i]\right] = \left[\left[\sum_{j=1}^n \partial_j, \theta\right], L_i\right] + \left[\left[L_i, \sum_{j=1}^n \partial_j\right], \theta\right]$$
$$= \left[\left(m^{-1} + n - 1\right)Id, L_i^m\right] + [0, \theta] = 0.$$

To prove that operators L_k and L_l, $k \neq l$, commute one can use the arguments of the paper [11]. Consider an involution δ on the space of all differential operators on R^n corresponding to the change $x \to -x$ and the standard anti-involution $*$: operator L^* is a formal adjoint to L. We have $[L_1^\delta, L_2^\delta] = [L_1, L_2]^\delta$ and $[L_1^*, L_2^*] = -[L_1, L_2]^*$. Operators L_k have the following properties with respect to these involutions: $L_k^* = L_k^\delta = (-1)^k L_k$. Now, consider the commutator $C = [L_k, L_l]$. By the Jacobi identity $[C, H] = 0$, therefore we can use Berezin's lemma [15] which states that in such a case the highest symbol of C is polynomial in x. In this case it is also periodic, hence the highest symbol must be constant. Now, we also have

$$C^\delta = [L_k, L_l]^\delta = [L_k^\delta, L_l^\delta] = [L_k^*, L_l^*] = -[L_k, L_l]^* = -C^*. \tag{16}$$

Since the highest symbol of C is constant the highest symbols of C^* and C^δ are the same but it follows from (16) they they must be different by a sign. This means that the highest symbol of C is zero and hence C is zero.

Theorem 3. *Deformed quantum CM problem (1) is integrable for all n and m and algebraically integrable for integer m.*

Proof. The complete family of the commuting quantum integrals for arbitrary m is given by the previous theorems. The algebraic integrability in the case when m is integer follows from the general result due to Chalykh, Etingof and Oblomkov (see Theorem 3.8 in [4]).

5 Trigonometric and rational degenerations

Trigonometric degenerations of the Weierstrass $\wp$-function corresponds to the case when one of the half periods ω_1 or ω_2 is infinite, which happens when two of the roots of the polynomial (3) collide. For example, the case of $e_1 = e_2 = a$ and $e_3 = -2a$ corresponds to $\omega = \infty$, $\tilde{\omega} = i\frac{\pi}{\sqrt{12a}}$ and $\wp(z) = a + \frac{3a}{\sinh^2 \sqrt{3a}z}$. Choosing $a = \frac{1}{3}$ we have

$$\wp(z) = \tfrac{1}{3} + \tfrac{1}{\sinh^2 z} = z^{-2} - \sum_{k=1}^{\infty} \tfrac{2^{2k+2}}{(2k+2)} \tfrac{B_{2k+2}}{(2k)!} z^{2k}, \quad \varsigma(z) = -\tfrac{1}{3}z + \coth z, \quad \text{and} \quad \gamma_{2k} = -\tfrac{2^{2k+2}}{(2k+2)} \tfrac{B_{2k+2}}{(2k)!},$$

where B_{2k+2} are the classical Bernoulli numbers defined by the expansion

$$\tfrac{z}{e^z-1} = 1 - \tfrac{1}{2}z + \sum_{k=1}^{\infty} \tfrac{B_{2k}}{(2k)!} z^{2k}.$$

In this case the Hamiltonian H takes the form

$$H = -\left(m\partial_1^2 + \partial_2^2 + \ldots + \partial_{n-1}^2 + \partial_n^2\right) + \tfrac{(m+1)(n-1)}{3}\left(1 - m + \tfrac{mn}{2}\right) + \sum_{k=2}^{n} \tfrac{2(m+1)}{\sinh^2(x_1-x_k)} + \sum_{2 \le j < k \le n} \tfrac{2m(m+1)}{\sinh^2(x_j-x_k)}.$$

Therefore, the formulae for the integrals in this case can be obtained using the following recursive formulae for constants $p_{2i,k}$:

$$p_{2i,k} = \tfrac{(1-m(k-2i-1))}{k-2i}p_{2i,k-1} + (1+m)\sum_{\substack{j=1 \\ j \ne 2}}^{i-1} C_{k-2j+1}^{k-2i} \tfrac{2^{2i-2j+2}}{(2i-2j+2)} B_{2i-2j+2} p_{2j-2,k-1}.$$

The integrability of this problem was shown in [2]. In [16] a recurrent formula was found for the quantum integrals with the highest symbols given by the deformed Newton sums. Our formulae correspond to the deformed elementary symmetric polynomials and seem to be new even in that degenerate case.

The rational degeneration corresponds to both periods be equal to infinity. In this case $\wp(z) = z^{-2}$ and all $\gamma_{2k} = 0$. Therefore, in this case only constants $p_{0,k}$, $k = 1, 2, \ldots$, are non zero.

6 Proof of Theorem 1.

We prove Theorem 1 by induction in n. For small n we showed this in the section 3. Now assume that the statement of the theorem is true for all $k < n$ and show that it is true for $k = n$.

Under this assumption, let us first show that commutator $[I, H]$ can be reduced to an expression on the additional terms X, $X_{\hat{j}}$ and $X_{\hat{k}\hat{l}}$.

Lemma 1.
$$[I, H] = [X, H] + \sum_{j=1}^{n} \left[X_{\hat{j}}\partial_j, 2\sum_{\substack{l=1 \\ l \ne j}}^{n} u_{jl}\right] + \sum_{1 \le k < l \le n} \left[X_{\hat{k}\hat{l}}\partial_k\partial_l, 2u_{kl}\right]. \tag{17}$$

Proof. Using (12) we obtain

$$[I, H] = \sum_{t=1}^{n-2} (-1)^{t+1} \sum_{\sigma \in \mathfrak{S}(t)} [I_{\hat{\sigma}}\partial_\sigma, H] + (-1)^n (n-1)[\partial_1 \ldots \partial_n, H] + [X, H],$$

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which is equal to

$$\sum_{j=1}^{n}\left[I_{\hat{j}}\partial_j, H_{\hat{j}} - \Delta_j + 2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right] + \sum_{t=2}^{n-2}(-1)^{t+1}\sum_{\sigma\in\mathfrak{S}(t)}\left[I_{\hat{\sigma}}\partial_\sigma, H_{\hat{\sigma}} - \Delta_\sigma + 2\sum_{j\in\sigma}\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk} - 2\sum_{\substack{j<k\\j,k\in\sigma}}u_{jk}\right]$$

$$+ (-1)^n(n-1)\left[\partial_1\ldots\partial_n, 2\sum_{1\leq j<k\leq n}u_{jk}\right] + [X, H].$$

Since $I_{\hat{\sigma}}$ commutes with $H_{\hat{\sigma}}$ (by the induction assumption) and with $-\Delta_\sigma$ (since it does not depend on x_i, $i\in\sigma$) we have

$$[I, H] = \sum_{j=1}^{n}\left[I_{\hat{j}}\partial_j, 2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right] + \sum_{t=2}^{n-2}(-1)^{t+1}\sum_{\sigma\in\mathfrak{S}(t)}\left[I_{\hat{\sigma}}\partial_\sigma, 2\sum_{j\in\sigma}\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk} - 2\sum_{\substack{j<k\\j,k\in\sigma}}u_{jk}\right]$$

$$+ (-1)^n(n-1)\left[\partial_1\ldots\partial_n, 2\sum_{1\leq j<k\leq n}u_{jk}\right] + [X, H].$$

Now we use (12) again and obtain

$$[I, H] = \sum_{j=1}^{n}\left[\sum_{t=1}^{n-3}(-1)^{t+1}\sum_{\sigma\in\mathfrak{S}(\hat{j};t)}I_{\hat{j}\hat{\sigma}}\partial_\sigma\partial_j, 2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right] + \sum_{j=1}^{n}\left[(-1)^{n-1}(n-2)\partial_1\ldots\partial_n + X_{\hat{j}}\partial_j, 2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right]$$

$$+ \sum_{t=2}^{n-2}(-1)^{t+1}\sum_{\sigma\in\mathfrak{S}(t)}\left[I_{\hat{\sigma}}\partial_\sigma, 2\sum_{j\in\sigma}\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk} - 2\sum_{\substack{j<k\\j,k\in\sigma}}u_{jk}\right]$$

$$+ (-1)^n(n-1)\left[\partial_1\ldots\partial_n, 2\sum_{1\leq j<k\leq n}u_{jk}\right] + [X, H].$$

If we cancel the repeated terms we obtain

$$[I, H] = \sum_{j=1}^{n}\left[(-1)^{n-1}(n-2)\partial_1\ldots\partial_n + X_{\hat{j}}\partial_j, 2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right] + \sum_{t=2}^{n-2}(-1)^{t}\sum_{\sigma\in\mathfrak{S}(t)}\left[I_{\hat{\sigma}}\partial_\sigma, 2\sum_{\substack{j<k\\j,k\in\sigma}}u_{jk}\right]$$

$$+ (-1)^n(n-1)\left[\partial_1\ldots\partial_n, 2\sum_{1\leq j<k\leq n}u_{jk}\right] + [X, H].$$

We use (12) again

$$
\begin{aligned}
[I,H] \;=\;& \sum_{j=1}^{n}\left[(-1)^{n-1}(n-2)\,\partial_1\ldots\partial_n+X_{\hat{j}}\partial_j,2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right]+\sum_{1\leq j<k\leq n}\left[\sum_{t=1}^{n-4}(-1)^{t+1}\sum_{\sigma\in\mathfrak{S}\left(\hat{j}\hat{k};t\right)}I_{\hat{\sigma}\hat{j}\hat{k}}\partial_\sigma\partial_j\partial_k,2u_{jk}\right]\\
&+\sum_{1\leq j<k\leq n}\left[(-1)^{n-2}(n-3)\,\partial_1\ldots\partial_n+X_{\hat{j}\hat{k}}\partial_j\partial_k,2u_{jk}\right]+\sum_{t=3}^{n-2}(-1)^{t}\sum_{\sigma\in\mathfrak{S}(t)}\left[I_{\hat{\sigma}}\partial_\sigma,2\sum_{\substack{j<k\\j,k\in\sigma}}u_{jk}\right]\\
&+(-1)^{n}(n-1)\left[\partial_1\ldots\partial_n,2\sum_{1\leq j<k\leq n}u_{jk}\right]+[X,H]\,.
\end{aligned}
$$

Finally, canceling the repeated terms, we get

$$
\begin{aligned}
[I,H] \;=\;& (-1)^{n}\,2\left[\partial_1\ldots\partial_n,-(n-2)\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}+(n-3)\sum_{1\leq j<k\leq n}u_{jk}+(n-1)\sum_{1\leq j<k\leq n}u_{jk}\right]\\
&+[X,H]+\sum_{j=2}^{n}\left[X_{\hat{j}}\partial_j,2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right]+\sum_{1\leq j<k\leq n}\left[X_{\hat{j}\hat{k}}\partial_j\partial_k,2u_{jk}\right],
\end{aligned}
$$

which simplifies to

$$
[I,H]=[X,H]+\sum_{j=1}^{n}\left[X_{\hat{j}}\partial_j,2\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}\right]+\sum_{1\leq j<k\leq n}\left[X_{\hat{j}\hat{k}}\partial_j\partial_k,2u_{jk}\right].
$$

Lemma 1 is proven.

Lemma 2. In the non deformed case

$$
[I_{\hat{1}},H_{\hat{1}}]=[X_{\hat{1}},H_{\hat{1}}]+\sum_{j=2}^{n}\left[X_{\hat{1}\hat{j}}\partial_j,2\sum_{\substack{l=2\\l\neq j}}^{n}u_{jl}\right]+\sum_{2\leq k<l\leq n}\left[X_{\hat{1}\hat{k}\hat{l}}\partial_k\partial_l,2u_{kl}\right]=0. \tag{18}
$$

Proof. To prove that

$$
[I_{\hat{1}},H_{\hat{1}}]=[X_{\hat{1}},H_{\hat{1}}]+\sum_{j=2}^{n}\left[X_{\hat{1}\hat{j}}\partial_j,2\sum_{\substack{l=2\\l\neq j}}^{n}u_{jl}\right]+\sum_{2\leq k<l\leq n}\left[X_{\hat{1}\hat{k}\hat{l}}\partial_k\partial_l,2u_{kl}\right]
$$

one must repeat the arguments of Lemma 1. We will need the addition theorem for the Weierstrass elliptic function [12]

$$
T_{ijk}\equiv\det\begin{pmatrix}\wp_{ij}&\wp_{jk}&\wp_{ki}\\\wp'_{ij}&\wp'_{jk}&\wp'_{ki}\\1&1&1\end{pmatrix}\equiv 0.
$$

Two cases must be considered to prove the lemma: n is odd and n is even. If n is even (18) is reduced to $2\sum_{j=2}^{n}\sum_{\substack{l=2 \\ l\neq j}}^{n}\left[X_{\hat{1}\hat{j}}\partial_j, u_{jl}\right] = 0$, as $X_{\hat{1}} = 0$ and $X_{\hat{1}\hat{k}\hat{l}} = 0$, $2 \leq k < l \leq n$. In this case we have

$$\sum_{j=2}^{n}\sum_{\substack{l=2 \\ l\neq j}}^{n}\left[X_{\hat{1}\hat{j}}\partial_j, u_{jl}\right] = \sum_{j=2}^{n}\sum_{\substack{l=2 \\ l\neq j}}^{n}X_{\hat{1}\hat{j}}\left[\partial_j, u_{jl}\right] = m\left(m+1\right)\sum_{j=2}^{n}\sum_{\substack{l=2 \\ l\neq j}}^{n}X_{\hat{1}\hat{j}}\wp'_{jl}$$

$$= m\left(m+1\right)\sum_{2\leq j<l\leq n}\wp'_{jl}\left(X_{\hat{1}\hat{j}} - X_{\hat{1}\hat{l}}\right) = m\left(m+1\right)\sum_{2\leq i<j<k\leq n}T_{ijk}X_{\hat{1}\hat{i}\hat{j}\hat{k}} = 0.$$

If n is odd (18) becomes $\left[X_{\hat{1}}, -\Delta_{\hat{1}}\right] + \sum_{2\leq k<l\leq n}\left[X_{\hat{1}\hat{k}\hat{l}}\partial_k\partial_l, 2u_{kl}\right] = 0$, since in this case $X_{\hat{1}\hat{j}} = 0$, $j = 2, \ldots, n$. We have

$$\left[X_{\hat{1}}, -\Delta_{\hat{1}}\right] + \sum_{2\leq k<l\leq n}\left[X_{\hat{1}\hat{k}\hat{l}}\partial_k\partial_l, 2u_{kl}\right] = \sum_{k=2}^{n}\frac{\partial^2 X_{\hat{1}}}{\partial x_k^2} + 2\sum_{k=2}^{n}\frac{\partial X_{\hat{1}}}{\partial x_k}\partial_k + 2\sum_{2\leq k<l\leq n}X_{\hat{1}\hat{k}\hat{l}}\left(u'_{kl}\partial_l - u'_{kl}\partial_k - u''_{kl}\right)$$

$$= \sum_{k=2}^{n}\frac{\partial^2 X_{\hat{1}}}{\partial x_k^2} - 2\sum_{2\leq k<l\leq n}X_{\hat{1}\hat{k}\hat{l}}u''_{kl} + 2\sum_{k=2}^{n}\left(\frac{\partial X_{\hat{1}}}{\partial x_k} - \sum_{\substack{l=2 \\ l\neq k}}^{n}X_{\hat{1}\hat{k}\hat{l}}u'_{kl}\right)\partial_k = 0.$$

Lemma 2 is proven.

Lemma 3. The operators $\mathfrak{D}^n$ satisfy the relation

$$\begin{aligned} &\left[\mathfrak{D}^n, \wp\left(x_1 - x_i\right)\right] + \left(1+m\right)\left[\left[\varsigma\left(x_1 - x_i\right), \mathfrak{D}^{n-1}\right], \wp\left(x_1 - x_i\right)\right] \\ &+m\left[\mathfrak{D}^{n-1}, \wp\left(x_1 - x_i\right)\right]\partial - \frac{(1+m)}{2}\wp'\left(x_1 - x_i\right)\mathfrak{D}^{n-1} - \frac{(1-m)}{2}\mathfrak{D}^{n-1}\wp'\left(x_1 - x_i\right) = 0 \end{aligned} \tag{19}$$

for any $i = 2, \ldots, n$, where $\partial = \frac{\partial}{\partial x_1}$.

Proof.

Let us denote the left hand side of the relation (19) by $\mathcal{Y}_n$.

It can be shown by a simple direct calculation that (19) is true if $n = 1, 2, 3, 4$. The equations in these cases are

$$n = 1 : \mathcal{Y}_1 = \left[\partial, \wp\right] + m\left[1, \wp\right]\partial - \frac{(1+m)}{2}\wp' - \frac{(1-m)}{2}\wp' = 0,$$

$$n = 2 : \mathcal{Y}_2 = \left[\frac{1-m}{2}\partial^2, \wp\right] + \left(1+m\right)\left[\left[\varsigma, \partial\right], \wp\right] + m\left[\partial, \wp\right]\partial - \frac{(1+m)}{2}\wp'\partial - \frac{(1-m)}{2}\partial\wp' = 0,$$

$$n = 3 : \mathcal{Y}_3 = \left[\frac{1-2m}{3}\partial^3, \wp\right] + \left(1+m\right)\left[\left[\varsigma, \partial^2\right], \wp\right] + m\left[\partial^2, \wp\right]\partial - \frac{(1+m)}{2}\wp'\partial^2 - \frac{(1-m)}{2}\partial^2\wp' = 0,$$

$$n = 4 : \mathcal{Y}_4 = \left[\frac{1-3m}{4}\partial^4, \wp\right] + \left(1+m\right)\left[\left[\varsigma, \partial^3\right], \wp\right] + m\left[\partial^3, \wp\right]\partial - \frac{(1+m)}{2}\wp'\partial^3 - \frac{(1-m)}{2}\partial^3\wp' = 0.$$

For $n \geq 5$ we use (6) to obtain

$$\begin{aligned} \mathcal{Y}_n = & \left[p_{0,n}\partial^n, \wp\right] + \left(1+m\right)\left[\left[\varsigma, p_{0,n-1}\partial^{n-1}\right], \wp\right] + m\left[p_{0,n-1}\partial^{n-1}, \wp\right]\partial - \frac{(1+m)}{2}\wp'p_{0,n-1}\partial^{n-1} \\ & -\frac{(1-m)}{2}p_{0,n-1}\partial^{n-1}\wp' + \left[\sum_{i=2}^{\left[\frac{n}{2}\right]}p_{2i,n}\partial^{n-2i}, \wp\right] + \left(1+m\right)\left[\left[\varsigma, \sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i}\right], \wp\right] \\ & +m\left[\sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i}, \wp\right]\partial - \frac{(1+m)}{2}\wp'\sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i} - \frac{(1-m)}{2}\sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i}\wp'. \end{aligned}$$

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We can express $p_{0,n}$ through $p_{0,n-1}$ and $p_{2i,n}$ through $p_{2l,n-1}$, $l = 0, \ldots 2i$, as described by (9) and obtain

$$
\begin{aligned}
\mathcal{Y}_n \;=\; & \left[\tfrac{1-m(n-1)}{n} p_{0,n-1}\partial^n, \wp\right] + (1+m)\left[\left[\varsigma, p_{0,n-1}\partial^{n-1}\right], \wp\right] + m\left[p_{0,n-1}\partial^{n-1}, \wp\right]\partial \\
& -\tfrac{(1+m)}{2}\wp' p_{0,n-1}\partial^{n-1} - \tfrac{(1-m)}{2} p_{0,n-1}\partial^{n-1}\wp' - (1+m)\left[\sum_{i=2}^{\left[\frac{n-1}{2}\right]}(2i-2)!C_{n-1}^{n-2i}\gamma_{2i-2}p_{0,n-1}\partial^{n-2i}, \wp\right] \\
& + \left[\sum_{i=2}^{\left[\frac{n-1}{2}\right]}\tfrac{(1-m(n-2i-1))}{n-2i}p_{2i,n-1}\partial^{n-2i}, \wp\right] + (1+m)\left[\left[\varsigma, \sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i}\right], \wp\right] \\
& + m\left[\sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i}, \wp\right]\partial - \tfrac{(1+m)}{2}\wp'\sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i} - \tfrac{(1-m)}{2}\sum_{i=2}^{\left[\frac{n-1}{2}\right]}p_{2i,n-1}\partial^{n-1-2i}\wp' \\
& - (1+m)\sum_{i=2}^{\left[\frac{n-1}{2}\right]-2}\left[\sum_{j=i+2}^{\left[\frac{n-1}{2}\right]}(2j-2i-2)!C_{n-2i-1}^{n-2j}\gamma_{2j-2i-2}p_{2i,n-1}\partial^{n-2j}, \wp\right].
\end{aligned}
$$

Denote

$$
\begin{aligned}
\mathcal{W}_k \;=\; & \left[\tfrac{1-m(k-1)}{k}\partial^k, \wp\right] + (1+m)\left[\left[\varsigma, \partial^{k-1}\right], \wp\right] + m\left[\partial^{k-1}, \wp\right]\partial \\
& -\tfrac{(1+m)}{2}\wp'\partial^{k-1} - \tfrac{(1-m)}{2}\partial^{k-1}\wp' - (1+m)\sum_{i=2}^{\left[\frac{k-1}{2}\right]}\left[(2i-2)!C_{k-1}^{k-2i}\gamma_{2i-2}\partial^{k-2i}, \wp\right],
\end{aligned}
$$

then

$$
\mathcal{Y}_5 = p_{0,4}\mathcal{W}_5 + p_{4,4}\mathcal{Y}_1, \quad \mathcal{Y}_6 = p_{0,5}\mathcal{W}_6 + p_{4,5}\mathcal{Y}_2,
$$

$$
\mathcal{Y}_7 = p_{0,6}\mathcal{W}_7 + p_{4,6}\mathcal{Y}_3 + p_{6,6}\mathcal{Y}_1, \quad \mathcal{Y}_8 = p_{0,7}\mathcal{W}_8 + p_{4,7}\mathcal{Y}_4 + p_{6,7}\mathcal{Y}_2
$$

and for $k \geq 5$

$$
\mathcal{Y}_{2k-1} \;=\; p_{0,2k-2}\mathcal{W}_{2k-1} + \sum_{i=2}^{k-3}p_{2i,2k-2}\mathcal{W}_{2k-1-2i} + p_{2k-4,2k-2}\mathcal{Y}_3 + p_{2k-2,2k-2}\mathcal{Y}_1,
$$

$$
\mathcal{Y}_{2k} \;=\; p_{0,2k-1}\mathcal{W}_{2k} + \sum_{i=2}^{k-3}p_{2i,2k-1}\mathcal{W}_{2k-2i} + p_{2k-4,2k-1}\mathcal{Y}_4 + p_{2k-2,2k-1}\mathcal{Y}_2.
$$

Below we show that $\mathcal{W}_n = 0$ for $n \geq 5$. Firstly,

$$
\begin{aligned}
\tfrac{10}{1+m}\mathcal{W}_5 \;=\; & 2\left[\partial^4, \wp\right]\partial - 3\partial^4\wp' + 10\left[\left[\varsigma, \partial^4\right], \wp\right] - 5\wp'\partial^4 - 80\left[\gamma_2\partial^1, \wp\right] + 2\partial\left[\partial^3, \wp\right]\partial \\
& -3\wp'\partial^4 - 3\partial^4\wp' + 10\partial\left[\left[\varsigma, \partial^3\right], \wp\right] + 10\left[\partial, \wp\right]\left[\varsigma, \partial^3\right] + 10\left[\varsigma, \partial\right]\left[\partial^3, \wp\right] - 80\left[\gamma_2\partial^1, \wp\right].
\end{aligned}
$$

We calculate that $\partial\left[\left[\varsigma, \partial^3\right], \wp\right] = -\tfrac{1}{4}\partial\left[\partial^3, \wp\right]\partial + \tfrac{1}{2}\partial\wp'\partial^3 + \tfrac{1}{4}\partial^4\wp'$, therefore

$$
\begin{aligned}
\tfrac{10}{1+m}\mathcal{W}_5 \;=\; & -\tfrac{1}{2}\partial\left[\partial^3, \wp\right]\partial - 3\wp'\partial^4 - \tfrac{1}{2}\partial^4\wp' + 5\partial\wp'\partial^3 + 10\wp'\left[\varsigma, \partial^3\right] + 10\wp\left[\partial^3, \wp\right] - 80\gamma_2\wp' \\
& -5\wp'''\partial^2 - \tfrac{5}{2}\wp^{(4)}\partial - \tfrac{1}{2}\wp^{(5)} - 80\gamma_2\wp' \\
& +10\wp'\left(3\wp\partial^2 + 3\wp'\partial + \wp''\right) + 10\wp\left(3\wp'\partial^2 + 3\wp''\partial + \wp'''\right) = 0.
\end{aligned}
$$

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In this calculation the identities which are the derivatives of the differential equation of the Weierstrass elliptic function (3) have been used. At the next step we use the induction assumption that for any $s < k$ $\mathcal{W}_s = 0$. Then $\mathcal{W}_k$ can be simplified into

$$
\begin{aligned}
\mathcal{W}_k \quad = \quad & \tfrac{1+m}{k}\left[\partial^{k-1}, \wp\right]\partial - \tfrac{(1+m)}{2}\tfrac{k-2}{k}\partial^{k-1}\wp' + (1+m)\left[\left[\varsigma, \partial^{k-1}\right], \wp\right] \\
& -\tfrac{(1+m)}{2}\wp'\partial^{k-1} - (1+m)\sum_{j=2}^{\left[\frac{k-1}{2}\right]}\left[(2j-2)!C_{k-1}^{k-2j}\gamma_{2j-2}\partial^{k-2j}, \wp\right].
\end{aligned}
$$

We consider

$$
\begin{aligned}
\tfrac{1}{1+m}\mathcal{W}_k \quad = \quad & \tfrac{1}{k}\left[\partial^{k-1}, \wp\right]\partial - \tfrac{k-2}{2k}\partial^{k-1}\wp' + \left[\left[\varsigma, \partial^{k-1}\right], \wp\right] - \tfrac{1}{2}\wp'\partial^{k-1} \\
& -\sum_{j=2}^{\left[\frac{k-1}{2}\right]}\left[(2j-2)!C_{k-1}^{k-2j}\gamma_{2j-2}\partial^{k-2j}, \wp\right].
\end{aligned}
$$

If $k = 2l$

$$
\begin{aligned}
\tfrac{1}{1+m}\mathcal{W}_{2l} \quad = \quad & \tfrac{1}{2l}\left[\partial^{2l-1}, \wp\right]\partial - \tfrac{l-1}{2l}\partial^{2l-1}\wp' + \left[\left[\varsigma, \partial^{2l-1}\right], \wp\right] - \tfrac{1}{2}\wp'\partial^{2l-1} \\
& -\sum_{j=2}^{l-1}\left[(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\partial^{2l-2j}, \wp\right]
\end{aligned}
$$

and if $k = 2l + 1$

$$
\begin{aligned}
\tfrac{1}{1+m}\mathcal{W}_{2l+1} \quad = \quad & \tfrac{1}{(2l+1)}\left[\partial^{2l}, \wp\right]\partial - \tfrac{2l-1}{2(2l+1)}\partial^{2l}\wp' + \left[\left[\varsigma, \partial^{2l}\right], \wp\right] - \tfrac{1}{2}\wp'\partial^{2l} \\
& -\sum_{j=2}^{l}\left[(2j-2)!C_{2l}^{2l+1-2j}\gamma_{2j-2}\partial^{2l+1-2j}, \wp\right].
\end{aligned}
$$

We show below that $\mathcal{W}_{2l} = 0$. The case of $\mathcal{W}_{2l+1}$ can be dealt with in the same way.

$$
\begin{aligned}
\tfrac{1}{1+m}\mathcal{W}_{2l} \quad = \quad & \tfrac{1}{2l}\left[\partial^{2l-1}, \wp\right]\partial - \tfrac{1}{2l}(l-1)\partial^{2l-1}\wp' + \left[\left[\varsigma, \partial^{2l-1}\right], \wp\right] \\
& -\tfrac{1}{2}\wp'\partial^{2l-1} - \sum_{j=2}^{l-1}\left[(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\partial^{2l-2j}, \wp\right] \\
= \quad & \tfrac{1}{2l}\left[\partial\partial^{2l-2}, \wp\right]\partial - \tfrac{l-1}{2l}\partial\partial^{2l-2}\wp' + \left[\left[\varsigma, \partial\partial^{2l-2}\right], \wp\right] \\
& -\tfrac{1}{2}\wp'\partial\partial^{2l-2} - \sum_{j=2}^{l-1}\left[(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\partial\partial^{2l-2j-1}, \wp\right] \\
= \quad & \tfrac{1}{2l}\partial\left[\partial^{2l-2}, \wp\right]\partial - \tfrac{l-1}{2l}\partial\partial^{2l-2}\wp' - \tfrac{l-1}{2l}\wp'\partial\partial^{2l-2} + \partial\left[\left[\varsigma, \partial^{2l-2}\right], \wp\right] \\
& +\left[\partial, \wp\right]\left[\varsigma, \partial^{2l-2}\right] + \left[\varsigma, \partial\right]\left[\partial^{2l-2}, \wp\right] \\
& -\partial\sum_{j=2}^{l-1}\left[(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\partial^{2l-2j-1}, \wp\right] - \sum_{j=2}^{l-1}(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\wp'\partial^{2l-2j-1}.
\end{aligned}
$$

From the inductive assumption we have that $\mathcal{W}_{2l-1} = 0$, then

$$
\begin{aligned}
\left[\left[\varsigma, \partial^{2l-2}\right], \wp\right] \quad = \quad & -\tfrac{1}{2l-1}\left[\partial^{2l-2}, \wp\right]\partial + \tfrac{2l-3}{2(2l-1)}\partial^{2l-2}\wp' + \tfrac{1}{2}\wp'\partial^{2l-2} \\
& +\sum_{j=2}^{l-1}\left[(2j-2)!C_{2l-2}^{2l-1-2j}\gamma_{2j-2}\partial^{2l-1-2j}, \wp\right]
\end{aligned}
$$

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and therefore

$$
\begin{aligned}
-\tfrac{1}{1+m}\mathcal{W}_{2l} \;=\;& \tfrac{1}{2l(2l-1)}\partial\left[\partial^{2l-2},\wp\right]\partial + \tfrac{1}{2l(2l-1)}\partial\partial^{2l-2}\wp' - \tfrac{1}{2}\wp''\partial^{2l-2} - \tfrac{1}{2l}\wp'\partial\partial^{2l-2} \\
&+\partial\sum_{j=2}^{l-1}\left[\tfrac{(2l-2)!}{(2l-2j)!}\gamma_{2j-2}\partial^{2l-1-2j},\wp\right] - \wp'\left[\varsigma,\partial^{2l-2}\right] - \wp\left[\partial^{2l-2},\wp\right] \\
&+\sum_{j=2}^{l-1}(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\wp'\partial^{2l-2j-1} \\
=\;& \tfrac{1}{2l(2l-1)}\partial\left[\partial^{2l-1},\wp\right] - \tfrac{1}{2}\wp''\partial^{2l-2} - \tfrac{1}{2l}\wp'\partial^{2l-1} - \wp'\left[\varsigma,\partial^{2l-2}\right] - \wp\left[\partial^{2l-2},\wp\right] \\
&+\partial\sum_{j=2}^{l-1}\tfrac{(2l-2)!}{(2l-2j)!}\gamma_{2j-2}\left[\partial^{2l-1-2j},\wp\right] + \sum_{j=2}^{l-1}(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\wp'\partial^{2l-2j-1}.
\end{aligned}
$$

The commutators can be rewritten as

$$
\begin{aligned}
\left[\partial^{2l-1},\wp\right] \;=\;& \sum_{k=0}^{2l-2}C_{2l-1}^{2l-1-k}\wp^{(2l-1-k)}\partial^{k}, \qquad \left[\partial^{2l-2},\wp\right]=\sum_{k=0}^{2l-3}C_{2l-2}^{2l-2-k}\wp^{(2l-2-k)}\partial^{k}, \\
\left[\varsigma,\partial^{2l-2}\right] \;=\;& \sum_{k=0}^{2l-3}C_{2l-2}^{2l-2-k}\wp^{(2l-3-k)}\partial^{k}, \qquad \left[\partial^{2l-1-2j},\wp\right]=\sum_{k=0}^{2l-2-2j}C_{2l-1-2j}^{2l-1-2j-k}\wp^{(2l-1-2j-k)}\partial^{k}
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
-\tfrac{1}{1+m}\mathcal{W}_{2l} \;=\;& \tfrac{1}{2l(2l-1)}\sum_{k=1}^{2l-1}C_{2l-1}^{2l-k}\wp^{(2l-k)}\partial^{k} + \tfrac{1}{2l(2l-1)}\sum_{k=0}^{2l-2}C_{2l-1}^{2l-1-k}\wp^{(2l-k)}\partial^{k} \\
&-\tfrac{1}{2}\wp''\partial^{2l-2} - \tfrac{1}{2l}\wp'\partial^{2l-1} - \sum_{k=0}^{2l-3}C_{2l-2}^{2l-2-k}\wp'\wp^{(2l-3-k)}\partial^{k} - \sum_{k=0}^{2l-3}C_{2l-2}^{2l-2-k}\wp\wp^{(2l-2-k)}\partial^{k} \\
&+\sum_{j=2}^{l-1}\tfrac{(2l-2)!}{(2l-2j)!}\gamma_{2j-2}\sum_{k=1}^{2l-1-2j}C_{2l-1-2j}^{2l-2j-k}\wp^{(2l-2j-k)}\partial^{k} \\
&+\sum_{j=2}^{l-1}\tfrac{(2l-2)!}{(2l-2j)!}\gamma_{2j-2}\sum_{k=0}^{2l-2-2j}C_{2l-1-2j}^{2l-1-2j-k}\wp^{(2l-2j-k)}\partial^{k} + \sum_{j=2}^{l-1}(2j-2)!C_{2l-1}^{2l-2j}\gamma_{2j-2}\wp'\partial^{2l-2j-1}
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
-\tfrac{1}{1+m}\mathcal{W}_{2l} \;=\;& \tfrac{1}{2l(2l-1)}\sum_{k=1}^{2l-1}C_{2l-1}^{2l-k}\wp^{(2l-k)}\partial^{k} + \tfrac{1}{2l(2l-1)}\sum_{k=0}^{2l-2}C_{2l-1}^{2l-1-k}\wp^{(2l-k)}\partial^{k} \\
&-\tfrac{1}{2}\wp''\partial^{2l-2} - \tfrac{1}{2l}\wp'\partial^{2l-1} - \sum_{k=0}^{2l-3}C_{2l-2}^{2l-2-k}\left(\wp'\wp^{(2l-3-k)}+\wp\wp^{(2l-2-k)}\right)\partial^{k} \\
&+\sum_{k=1}^{2l-5}\sum_{j=2}^{l-1-\left[\frac{k}{2}\right]}\tfrac{(2l-2)!}{(2l-2j)!}\gamma_{2j-2}C_{2l-1-2j}^{2l-2j-k}\wp^{(2l-2j-k)}\partial^{k} \\
&+\sum_{k=0}^{2l-6}\sum_{j=2}^{l-1-\left[\frac{k+1}{2}\right]}\tfrac{(2l-2)!}{(2l-2j)!}\gamma_{2j-2}C_{2l-1-2j}^{2l-1-2j-k}\wp^{(2l-2j-k)}\partial^{k} + \sum_{k=1}^{l-2}(2l-2k-2)!C_{2l-1}^{2k}\gamma_{2l-2k-2}\wp'\partial^{2k-1}.
\end{aligned}
$$

It is shown below that the coefficients by all ∂^k, $k = 0, 1, \ldots 2l - 1$, are zero. We have

$$\partial^{2l-1}: \quad \frac{1}{2l(2l-1)}C^1_{2l-1}\wp' - \frac{1}{2l}\wp' = 0,$$

$$\partial^{2l-2}: \quad \frac{1}{2l(2l-1)}C^2_{2l-1}\wp'' + \frac{1}{2l(2l-1)}C^1_{2l-1}\wp'' - \frac{1}{2}\wp'' = 0,$$

$$\partial^{2l-3}: \quad \begin{aligned} &\frac{1}{2l(2l-1)}C^3_{2l-1}\wp^{(3)} + \frac{1}{2l(2l-1)}C^2_{2l-1}\wp^{(3)} - 2C^1_{2l-2}\wp'\wp \\ &= \left(\frac{12}{2l(2l-1)}C^3_{2l-1} + \frac{12}{2l(2l-1)}C^2_{2l-1} - 2C^1_{2l-2}\right)\wp'\wp = 0, \end{aligned}$$

$$\partial^{2l-4}: \quad \begin{aligned} &\frac{1}{2l(2l-1)}C^4_{2l-1}\wp^{(4)} + \frac{1}{2l(2l-1)}C^3_{2l-1}\wp^{(4)} - C^2_{2l-2}\left(\wp'\wp' + \wp\wp^{(2)}\right) \\ &= \left(\frac{12}{2l(2l-1)}C^4_{2l-1} + \frac{12}{2l(2l-1)}C^3_{2l-1} - C^2_{2l-2}\right)\left(\wp'\wp' + \wp\wp^{(2)}\right) = 0. \end{aligned}$$

When $k = 2l - 2q - 1$, $q = 2, \ldots, l - 1$, coefficient by $\partial^{2l-2q-1}$ is

$$K_{2l-2q-1} = \frac{(2l-2)!}{(2l-2q-1)!}\left(\frac{\wp^{(2q+1)}}{(2q+1)!} - \frac{\wp'\wp^{(2q-2)} + \wp\wp^{(2q-1)}}{(2q-1)!} + \sum_{j=2}^{q-1}\frac{\gamma_{2j-2}}{(2q+1-2j)!}\wp^{(2q+1-2j)} + \frac{2q}{2q-1}\gamma_{2q-2}\wp'\right).$$

To show that this coefficient is zero let us calculate its Laurent expansion and show that it consists of the terms by the positive degrees of z only. Since it is also a doubly periodic function it can only be zero. The Laurent expansion for the derivatives of the Weierstrass function are given by

$$\begin{aligned} \wp' &= -2z^{-3} + O(z), \\ \wp^{(2q+1)} &= -(2q+2)!z^{-(2q+3)} + O(z), \\ \wp^{(2q+1-2j)} &= -(2q+2-2j)!z^{-(2q-2j+3)} + O(z). \end{aligned}$$

Therefore, we can find that

$$\begin{aligned} \wp\wp^{(2q-1)} &= -(2q)!z^{-(2q+3)} - (2q)!\sum_{i=1}^{q-1}\gamma_{2i}z^{-(2q+1-2i)} + O(z), \\ \wp'\wp^{(2q-2)} &= -2(2q-1)!z^{-(2q+3)} - 2(2q-2)!\gamma_{2q-2}z^{-3} + (2q-1)!\sum_{i=1}^{q-1}2i\gamma_{2i}z^{-(2q+1-2i)} + O(z) \end{aligned}$$

and

$$-\frac{\left(\wp'\wp^{(2q-2)} + \wp\wp^{(2q-1)}\right)}{(2q-1)!} = (2q+2)z^{-(2q+3)} + \sum_{i=1}^{q-2}(2q-2i)\gamma_{2i}z^{-(2q+1-2i)} + \frac{4q}{2q-1}\gamma_{2q-2}z^{-3} + O(z).$$

Therefore

$$\begin{aligned} K_{2l-2q-1} &= \frac{(2l-2)!}{(2l-2q-1)!}\left(-(2q+2)z^{-(2q+3)} + (2q+2)z^{-(2q+3)} + \sum_{i=1}^{q-2}(2q-2i)\gamma_{2i}z^{-(2q+1-2i)}\right. \\ &\qquad\qquad\left. + \frac{4q}{2q-1}\gamma_{2q-2}z^{-3} - \sum_{j=2}^{q-1}\gamma_{2j-2}(2q+2-2j)z^{-(2q-2j+3)} - \gamma_{2q-2}\frac{4q}{2q-1}z^{-3} + O(z)\right) \\ &= \frac{(2l-2)!}{(2l-2q-1)!}(O(z)) = 0. \end{aligned}$$

When $k = 2l - 2q; q = 3, ..., l-1$, coefficient by ∂^{2l-2q} is

$$K_{2l-2q} = \frac{(2l-2)!}{(2l-2q)!} \left(\frac{1}{(2q)!} \wp^{(2q)} - \frac{\left(\wp' \wp^{(2q-3)} + \wp\wp^{(2q-2)} \right)}{(2q-2)!} + \sum_{j=2}^{q-1} \frac{1}{(2q-2j)!} \gamma_{2j-2} \wp^{(2q-2j)} \right).$$

The Laurent expansion for the derivatives of the Weierstrass function are given by

$$\wp^{(2q)} = (2q+1)! z^{-(2q+2)} + (2q)! \gamma_{2q} + O\left(z^2\right),$$

$$\wp^{(2q-2j)} = (2q-2j+1)! z^{-(2q-2j+2)} + (2q-2j)! \gamma_{2q-2j} + O\left(z^2\right)$$

and we calculate that

$$\wp\wp^{(2q-2)} = (2q-1)! z^{-(2q+2)} + (2q-2)! 2q\gamma_{2q-2} z^{-2} + (2q-1)! (q+1) \gamma_{2q}$$
$$+ (2q-1)! \left(\sum_{i=1}^{q-2} \gamma_{2i} z^{-(2q-2i)} \right) + O\left(z^2\right),$$

$$\wp'\wp^{(2q-3)} = 2(2q-2)! z^{-(2q+2)} - (2q-2)! 2q\gamma_{2q-2} z^{-2} - (2q-2)! (2q) \left(\frac{2q+2}{3} \right) \gamma_{2q}$$
$$- (2q-2)! \sum_{i=1}^{q-2} 2i\gamma_{2i} z^{-(2q-2i)} + O\left(z^2\right)$$

and

$$-\frac{\left(\wp\wp^{(2q-2)} + \wp'\wp^{(2q-3)} \right)}{(2q-2)!} = -(2q+1) z^{-(2q+2)} - \sum_{i=1}^{q-2} (2q-1-2i) \gamma_{2i} z^{-(2q-2i)} - \left(\frac{(1+q)(2q-3)}{3} \right) \gamma_{2q} + O\left(z^2\right).$$

Therefore

$$K_{2l-2q} = \frac{(2l-2)!}{(2l-2q)!} \left((2q+1) z^{-(2q+2)} + \gamma_{2q} - (2q+1) z^{-(2q+2)} - \sum_{i=1}^{q-2} (2q-1-2i) \gamma_{2i} z^{-(2q-2i)} \right.$$

$$\left. - \left(\frac{(1+q)(2q-3)}{3} \right) \gamma_{2q} + \sum_{j=2}^{q-1} \gamma_{2j-2} (2q-2j+1) z^{-(2q-2j+2)} + \sum_{j=2}^{q-1} \gamma_{2j-2}\gamma_{2q-2j} + O\left(z^2\right) \right)$$

$$= \frac{(2l-2)!}{(2l-2q)!} \left(-\frac{(2q+3)(q-2)}{3} \gamma_{2q} + \sum_{j=2}^{q-1} \gamma_{2j-2}\gamma_{2q-2j} + O\left(z^2\right) \right) = 0.$$

Here we have used the identity (5).

The last case to consider is when $k = 0$. In this case we need to show that

$$K_0 = \frac{1}{2l(2l-1)} \wp^{(2l)} - \left(\wp'\wp^{(2l-3)} + \wp\wp^{(2l-2)} \right) + \sum_{j=2}^{l-1} \frac{(2l-2)!}{(2l-2j)!} \gamma_{2j-2} \wp^{(2l-2j)}$$

is zero. We have

$$
\begin{aligned}
K_0 &= (2l-2)!\left((2l+1)\,z^{-(2l+2)} + \gamma_{2l} - (2l+1)\,z^{-(2l+2)} - \sum_{i=1}^{l-2}(2l-1-2i)\,\gamma_{2i}z^{-(2l-2i)} \right.\\[2mm]
&\qquad\qquad \left. -\left(\tfrac{(1+l)(2l-3)}{3}\right)\gamma_{2l} + \sum_{j=2}^{l-1}\gamma_{2j-2}\,(2l-2j+1)\,z^{-(2l-2j+2)} + \sum_{j=2}^{l-1}\gamma_{2j-2}\gamma_{2l-2j} + O\left(z^2\right) \right)\\[2mm]
&= (2l-2)!\left(-\tfrac{(2l+3)(l-2)}{3}\gamma_{2l} + \sum_{j=2}^{l-1}\gamma_{2j-2}\gamma_{2l-2j} + O\left(z^2\right)\right) = 0.
\end{aligned}
$$

The proof of Lemma 3 is now finished.

Lemma 4. The following identity holds

$$
\left[\Theta,\sum_{j=2}^{n} u_{1j}\right] + m\sum_{j=2}^{n}[\Theta_{\hat{j}},u_{1j}]\,\partial - \tfrac{(1+m)}{2}\sum_{j=2}^{n}u_{1j}'\Theta_{\hat{j}} - \tfrac{(1-m)}{2}\sum_{j=2}^{n}\Theta_{\hat{j}}u_{1j}' + m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}[\Theta_{\hat{k}\hat{l}},u_{1k}]\,u_{1l} = 0.
$$

Here again $\partial = \frac{\partial}{\partial x_1}$.

Proof. From Lemma 3

$$
[\mathfrak{D}^n,u_{1j}] + \left[[\varsigma_j,\mathfrak{D}^{n-1}],u_{1j}\right] + m\left[\mathfrak{D}^{n-1},u_{1j}\right]\partial - \tfrac{(1+m)}{2}u_{1j}'\mathfrak{D}^{n-1} - \tfrac{(1-m)}{2}\mathfrak{D}^{n-1}u_{1j}' = 0. \tag{20}
$$

Note also that

$$
\begin{aligned}
\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t\left([\mathfrak{D}^{n-t-1},u_{1j}]\,\partial\right) &= \sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t\left([\mathfrak{D}^{n-t-1},u_{1j}]\right)\partial\\[2mm]
&\quad + \sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{k}\hat{l};t)}\left[ad_{\varsigma_\sigma}^t\left(\mathfrak{D}^{n-2-t}\right),u_{1k}\right]u_{1l}.
\end{aligned}
$$

We sum (20) and then use (11) to obtain Θ and $\Theta_{\hat{k}}^n$. We have

$$
\begin{aligned}
0 &= \sum_{t=0}^{\left[\frac{n}{2}\right]}\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t\left([\mathfrak{D}^{n-t},u_{1j}] + \left[[\varsigma_j,\mathfrak{D}^{n-t-1}],u_{1j}\right] + m\left[\mathfrak{D}^{n-t-1},u_{1j}\right]\partial \right.\\[2mm]
&\qquad\qquad\qquad\qquad\qquad \left. - \tfrac{(1+m)}{2}u_{1j}'\mathfrak{D}^{n-t-1} - \tfrac{(1-m)}{2}\mathfrak{D}^{n-t-1}u_{1j}' \right)\\[2mm]
&= \sum_{t=0}^{\left[\frac{n}{2}\right]}\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)}\left[ad_{\varsigma_\sigma}^t\left(\mathfrak{D}^{n-t}\right),u_{1j}\right] + \sum_{t=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)}\left[ad_{\varsigma_{\{j,\sigma\}}}^t\left(\mathfrak{D}^{n-t-1}\right),u_{1j}\right]\\[2mm]
&\quad + m\sum_{t=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t\left([\mathfrak{D}^{n-t-1},u_{1j}]\right)\partial + m\sum_{t=0}^{\left[\frac{n-2}{2}\right]}\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{k}\hat{l};t)}\left[ad_{\varsigma_\sigma}^t\left(\mathfrak{D}^{n-2-t}\right),u_{1k}\right]u_{1l}\\[2mm]
&\quad - \tfrac{(1+m)}{2}\sum_{t=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)} u_{1j}'ad_{\varsigma_\sigma}^t\left(\mathfrak{D}^{n-t-1}\right) - \tfrac{(1-m)}{2}\sum_{t=0}^{\left[\frac{n-1}{2}\right]}\sum_{j=2}^{n}\sum_{\sigma\in\mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t\left(\mathfrak{D}^{n-t-1}\right)u_{1j}'
\end{aligned}
$$

$$
= \sum_{t=0}^{\left[\frac{n}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1};t)} \left[ad_{\varsigma_\sigma}^t \left(\mathfrak{D}^{n-t} \right), \sum_{j=2}^n u_{1j} \right] + m \sum_{t=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=2}^n \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t \left(\left[\mathfrak{D}^{n-t-1}, u_{1j} \right] \right) \partial
$$

$$
+ m \sum_{t=0}^{\left[\frac{n-2}{2}\right]} \sum_{k=2}^n \sum_{\substack{l=2 \\ l \neq k}}^n \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{k}\hat{l};t)} \left[ad_{\varsigma_\sigma}^t \left(\mathfrak{D}^{n-2-t} \right), u_{1k} \right] u_{1l}
$$

$$
- \frac{(1+m)}{2} \sum_{t=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=2}^n \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{j};t)} u'_{1j} ad_{\varsigma_\sigma^\mu}^t \left(\mathfrak{D}^{n-t-1} \right) - \frac{(1-m)}{2} \sum_{t=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=2}^n \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{j};t)} ad_{\varsigma_\sigma}^t \left(\mathfrak{D}^{n-t-1} \right) u'_{1j}
$$

$$
= \left[\Theta, \sum_{j=2}^n u_{1j} \right] + m \sum_{j=2}^n \left[\Theta_{\hat{j}}, u_{1j} \right] \partial - \frac{(1+m)}{2} \sum_{j=2}^n u'_{1j} \Theta_{\hat{j}} - \frac{(1-m)}{2} \sum_{j=2}^n \Theta_{\hat{j}} u'_{1j} + m \sum_{k=2}^n \sum_{\substack{l=2 \\ l \neq k}}^n \left[\Theta_{\hat{k}\hat{l}}, u_{1k} \right] u_{1l}.
$$

Lemma 4 is proven.

Lemma 5. The following identity holds

$$
[\Delta, \Theta] = 2 \sum_{k=2}^n \left[u_{1k}, \Theta_{\hat{k}} \right] \partial_k - 2m \left(\sum_{k=2}^n \left[u_{1k}, \Theta_{\hat{k}} \right] \right) \partial_1 - (1+m) \sum_{k=2}^n \left[u'_{1k}, \Theta_{\hat{k}} \right] + m \sum_{k=2}^n \sum_{\substack{l=2 \\ l \neq k}}^n \left[u_{1k}, \left[u_{1l}, \Theta_{\hat{k}\hat{l}} \right] \right]. \quad (21)
$$

Proof.
$$
[\Delta, \Theta] = 2m \frac{\partial \Theta}{\partial x_1} \partial_1 + 2 \sum_{k=2}^n \frac{\partial \Theta}{\partial x_k} \partial_k + m \frac{\partial^2 \Theta}{\partial x_1^2} + \sum_{k=2}^n \frac{\partial^2 \Theta}{\partial x_k^2}.
$$

From (11) it follows that $\frac{\partial \Theta}{\partial x_k} = \left[u_{1k}, \Theta_{\hat{k}} \right]$ and $\frac{\partial \Theta}{\partial x_1} = - \sum_{k=2}^n \left[u_{1k}, \Theta_{\hat{k}} \right]$, therefore $\frac{\partial^2 \Theta}{\partial x_k^2} = - \left[u'_{1k}, \Theta_{\hat{k}} \right]$ and $\frac{\partial^2 \Theta}{\partial x_1^2} = - \sum_{k=2}^n \left[u'_{1k}, \Theta_{\hat{k}} \right] + \sum_{k=2}^n \sum_{\substack{l=2 \\ l \neq k}}^n \left[u_{1k}, \left[u_{1l}, \Theta_{\hat{k}\hat{l}} \right] \right]$, and we arrive at (21).

Lemma 6.

$$
R = [\Theta, H] + \sum_{j=2}^n \left[\Theta_{\hat{j}} \partial_j, 2 \sum_{\substack{l=1 \\ l \neq j}}^n u_{jl} \right] + \sum_{2 \leq k < l \leq n} \left[\Theta_{\hat{k}\hat{l}}, 2 \left(u_{1k} + u_{1l} \right) \right] X_{\{k,l\}} = 0.
$$

Proof. We have

$$
R = [\Delta, \Theta] + 2 \sum_{k=2}^n \left[\Theta, u_{1k} \right] + 2 \sum_{j=2}^n \sum_{\substack{l=1 \\ l \neq j}}^n \left[\Theta_{\hat{j}}, u_{jl} \right] \partial_j + 2 \sum_{j=2}^n \sum_{\substack{l=1 \\ l \neq j}}^n \Theta_{\hat{j}} \left[\partial_j, u_{jl} \right] + 2 \sum_{2 \leq k < l \leq n} \left[\Theta_{\hat{k}\hat{l}}, u_{1k} + u_{1l} \right] u_{kl}
$$

Using Lemma 5 R is rewritten as

$$
R = 2 \sum_{k=2}^n \left[u_{1k}, \Theta_{\hat{k}} \right] \partial_k - 2m \sum_{k=2}^n \left[u_{1k}, \Theta_{\hat{k}} \right] \partial_1 - (1+m) \sum_{k=2}^n \left[u'_{1k}, \Theta_{\hat{k}} \right] + m \sum_{k=2}^n \sum_{\substack{l=2 \\ l \neq k}}^n \left[u_{1k}, \left[u_{1l}, \Theta_{\hat{k}\hat{l}} \right] \right]
$$

$$
+ 2 \sum_{k=2}^n \left[\Theta, u_{1k} \right] + 2 \sum_{j=2}^n \left[\Theta_{\hat{j}}, u_{1j} \right] \partial_j + 2 \sum_{j=2}^n \sum_{\substack{l=1 \\ l \neq j}}^n \Theta_{\hat{j}} u'_{jl} + 2 \sum_{2 \leq k < l \leq n} \left[\Theta_{\hat{k}\hat{l}}, u_{1k} + u_{1l} \right] u_{kl}
$$

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and, hence,

$$
\begin{aligned}
R &= -2m\sum_{k=2}^{n}\left[u_{1k},\Theta_{\hat{k}}\right]\partial-(1+m)\sum_{k=2}^{n}\left[u'_{1k},\Theta_{\hat{k}}\right]+m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\left[u_{1k},\left[u_{1l},\Theta_{\hat{k}\hat{l}}\right]\right]+2\sum_{k=2}^{n}\left[\Theta,u_{1k}\right]\\
&\quad -2\sum_{j=2}^{n}\Theta_{\hat{j}}u'_{1j}+2\sum_{2\leq k<l\leq n}\left(\Theta_{\hat{k}}-\Theta_{\hat{l}}\right)u'_{kl}+2\sum_{2\leq k<l\leq n}\left[\Theta_{\hat{k}\hat{l}},u_{1k}+u_{1l}\right]u_{kl}.
\end{aligned}
$$

Now, let us apply Lemma 4 which simplifies the above expression into the following

$$
\begin{aligned}
R &= (1+m)\sum_{k=2}^{n}u'_{1k}\Theta_{\hat{k}}+(1-m)\sum_{k=2}^{n}\Theta_{\hat{k}}u'_{1k}-2m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\left[\Theta_{\hat{k}\hat{l}},u_{1k}\right]u_{1l}-(1+m)\sum_{k=2}^{n}u'_{1k}\Theta_{\hat{k}}\\
&\quad +(1+m)\sum_{k=2}^{n}\Theta_{\hat{k}}u'_{1k}+m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\left[u_{1k},\left[u_{1l},\Theta_{\hat{k}\hat{l}}\right]\right]+\sum_{2\leq k<l\leq n}\left[\Theta_{\hat{k}\hat{l}},2\left(u_{1k}+u_{1l}\right)\right]u_{kl}\\
&\quad -2\sum_{j=2}^{n}\Theta_{\hat{j}}u'_{1j}+2\sum_{2\leq k<l\leq n}\left(\Theta_{\hat{k}}-\Theta_{\hat{l}}\right)u'_{kl}\\
&= -2m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\left[\Theta_{\hat{k}\hat{l}},u_{1k}\right]u_{1l}+m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\left[u_{1k},\left[u_{1l},\Theta_{\hat{k}\hat{l}}\right]\right]\\
&\quad +2\sum_{2\leq k<l\leq n}\left[\Theta_{\hat{k}\hat{l}},\left(u_{1k}+u_{1l}\right)\right]u_{kl}+2\sum_{2\leq k<l\leq n}\left(\Theta_{\hat{k}}-\Theta_{\hat{l}}\right)u'_{kl}.
\end{aligned}
$$

From (11) $\Theta_{\hat{k}}-\Theta_{\hat{l}}=\left[\varsigma_{l}-\varsigma_{k},\Theta_{\hat{k}\hat{l}}\right]$ and, therefore, R is simplified to

$$
\begin{aligned}
R &= m\sum_{k=2}^{n}\sum_{\substack{l=2\\l\neq k}}^{n}\left[u_{1k}u_{1l},\Theta_{\hat{k}\hat{l}}\right]+2\sum_{2\leq k<l\leq n}\left[\Theta_{\hat{k}\hat{l}},\left(u_{1k}+u_{1l}\right)u_{kl}\right]+2\sum_{2\leq k<l\leq n}\left[\varsigma_{1l}-\varsigma_{1k},\Theta_{\hat{k}\hat{l}}\right]u'_{kl}\\
&= 2\sum_{2\leq k<l\leq n}\left[\left(\varsigma_{l}-\varsigma_{k}\right)u'_{kl}-\left(u_{1k}+u_{1l}\right)u_{kl}+mu_{1k}u_{1l},\Theta_{\hat{k}\hat{l}}\right].
\end{aligned}
$$

All derivatives with respect to x_1 of the term

$$
\left(\varsigma_{l}-\varsigma_{k}\right)u'_{kl}-\left(u_{1k}+u_{1l}\right)u_{kl}+mu_{1k}u_{1l}
$$

$$
= m\left(m+1\right)^{2}\left(\left(\varsigma\left(x_{1}-x_{l}\right)-\varsigma\left(x_{1}-x_{k}\right)\right)\wp'_{kl}-\left(\wp_{1k}+\wp_{1l}\right)\wp_{kl}+\wp_{1k}\wp_{1l}\right)
$$

are zero. Indeed, this follows from the fact that the first derivative is the addition theorem for the Weierstrass $\wp$-function.

$$
\frac{\partial}{\partial x_{1}}\left(\left(\varsigma\left(x_{1}-x_{l}\right)-\varsigma\left(x_{1}-x_{k}\right)\right)\wp'_{kl}-\left(\wp_{1k}+\wp_{1l}\right)\wp_{kl}+\wp_{1k}\wp_{1l}\right)
$$

$$
= \left(\wp_{1k}-\wp_{1l}\right)\wp'_{kl}-\left(\wp'_{1k}+\wp'_{1l}\right)\wp_{kl}+\wp'_{1k}\wp_{1l}+\wp_{1k}\wp'_{1l}=0.
$$

Therefore $R=2\sum_{2\leq k<l\leq n}\left[\left(\varsigma_{1l}-\varsigma_{1k}\right)u'_{kl}-\left(u_{1k}+u_{1l}\right)u_{kl}+mu_{1k}u_{1l},\Theta_{\hat{k}\hat{l}}\right]=0$ and Lemma 6 is proven.

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Proof of Theorem 1.

We use Lemma 3 to reduce the commutator $[I, H]$ to the expression (17) and we show below that (17) is zero. From (13)

$$X = \Theta + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1};2t)} X_\sigma \Theta_{\hat{\sigma}},$$

therefore we can rewrite (17) as following

$$
\begin{aligned}
[I, H] \quad = \quad & \left[X_{\hat{1}}\partial_1, 2\sum_{l=2}^{n} u_{1l}\right] + \left[\sum_{k=2}^{n} X_{\hat{1}\hat{k}}\partial_1\partial_k, 2u_{1k}\right] + \left[\Theta + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1};2t)} X_\sigma \Theta_{\hat{\sigma}}, H\right] \\
& + \left[\sum_{j=2}^{n} \left(\Theta_{\hat{j}} + \sum_{t=1}^{\left[\frac{n-3}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{j};2t)} X_\sigma \Theta_{\hat{j}\hat{\sigma}}\right) \partial_j, 2\sum_{\substack{l=1 \\ l \neq j}}^{n} u_{jl}\right] \\
& + \left[\sum_{2 \leq i < j \leq n} \left(\Theta_{\hat{i}\hat{j}} + \sum_{t=1}^{\left[\frac{n-4}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{i}\hat{j};2t)} X_\sigma \Theta_{\hat{i}\hat{j}\hat{\sigma}}\right) \partial_i\partial_j, 2u_{ij}\right] \\
= \quad & [\Theta, H] + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1};2t)} [X_\sigma, -\Delta_\sigma]\Theta_{\hat{\sigma}} + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1};2t)} \left[\Theta_{\hat{\sigma}}, -\Delta_{\hat{\sigma}} + 2\sum_{j=2}^{n} u_{1j}\right] X_\sigma \\
& + \sum_{j=2}^{n} \left[\Theta_{\hat{j}}\partial_j, 2\sum_{\substack{l=1 \\ l \neq j}}^{n} u_{jl}\right] + \sum_{j=2}^{n} \sum_{t=1}^{\left[\frac{n-3}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{j};2t)} \left[X_\sigma \partial_j, 2\sum_{l \in \sigma} u_{jl}\right] \Theta_{\hat{j}\hat{\sigma}} \\
& + \sum_{j=2}^{n} \sum_{t=1}^{\left[\frac{n-3}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{j};2t)} \left[\Theta_{\hat{j}\hat{\sigma}}\partial_j, 2\sum_{\substack{l=1 \\ l \notin j \cup \sigma}}^{n} u_{jl}\right] X_\sigma + \left[\sum_{2 \leq i < j \leq n} \partial_i\partial_j, 2u_{ij}\right] \Theta_{\hat{i}\hat{j}} \\
& + \sum_{2 \leq i < j \leq n} \sum_{t=1}^{\left[\frac{n-4}{2}\right]} \sum_{\sigma \in \mathfrak{S}(\hat{1}\hat{i}\hat{j};2t)} [X_\sigma \partial_i\partial_j, 2u_{ij}]\Theta_{\hat{i}\hat{j}\hat{\sigma}} + \left[X_{\hat{1}}\partial_1, 2\sum_{l=2}^{n} u_{1l}\right] + \left[\sum_{k=2}^{n} X_{\hat{1}\hat{k}}\partial_1\partial_k, 2u_{1k}\right]
\end{aligned}
$$

From the formulae for the system with $n = 2$ we have

$$\left[X_{\{k,l\}}, -\Delta_{\{k,l\}}\right] = -\left[\partial_k\partial_l, 2u_{kl}\right], \qquad \text{and} \qquad [\partial_1\partial_k, 2u_{1k}] = -\left[\Theta_{\{1,k\}}, H_{\{1,k\}}\right]$$

and from Lemma 2 we have that for any $\sigma' \in \mathfrak{S}\left(\hat{1}; 2t\right)$, $t \geq 2$,

$$[X_{\sigma'}, -\Delta_{\sigma'}] = -2 \sum_{\substack{k < l \\ k,l \in \sigma'}} \left[X_{\sigma' \setminus \{k,l\}}\partial_k\partial_l, u_{kl}\right]$$

and

$$\sum_{t=1}^{\left[\frac{n-3}{2}\right]} \sum_{j=2}^{n} \sum_{\sigma \in \mathfrak{S}(\sigma' \setminus \{j\};2t)} \left[X_\sigma \partial_j, 2\sum_{l \in \sigma} u_{jl}\right] \Theta_{\hat{j}\hat{\sigma}} = 0.$$

Hence, $[I, H]$ can be simplified to

$$
\begin{aligned}
[I,H] &= [\Theta, H] + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}\left(\hat{1};2t\right)} [\Theta_{\hat{\sigma}}, H_{\hat{\sigma}}] X_{\sigma} + \sum_{t=1}^{\left[\frac{n-2}{2}\right]} \sum_{\sigma \in \mathfrak{S}\left(\hat{1};2t\right)} \left[\Theta_{\hat{\sigma}}, 2\sum_{j \in \sigma} u_{1j}\right] X_{\sigma} \\
&+ \sum_{j=2}^{n} \left[\Theta_{\hat{j}}\partial_j, 2\sum_{\substack{l=1 \\ l\neq j}}^{n} u_{jl}\right] + \sum_{j=2}^{n} \sum_{t=1}^{\left[\frac{n-3}{2}\right]} \sum_{\sigma \in \mathfrak{S}\left(\hat{1}\hat{j};2t\right)} \left[\Theta_{\hat{j}\hat{\sigma}}\partial_j, 2\sum_{\substack{l=1 \\ l\notin j\cup\sigma}}^{n} u_{jl}\right] X_{\sigma} \\
&+ \left[X_{\hat{1}}\partial_1, 2\sum_{l=2}^{n} u_{1l}\right] - \sum_{k=2}^{n} \left[\Theta_{\{1,k\}}, H_{\{1,k\}}\right] X_{\hat{1}\hat{k}}.
\end{aligned}
$$

This expression is equal zero. To verify this we should use relation (14), apply Lemma 6, which gives us

$$
[\Theta, H] = -\sum_{j=2}^{n} \left[\Theta_{\hat{j}}\partial_j, 2\sum_{\substack{l=1 \\ l\neq j}}^{n} u_{jl}\right] - \sum_{2\leq k<l\leq n} \left[\Theta_{\hat{k}\hat{l}}, 2\left(u_{1k}+u_{1l}\right)\right] X_{\{k,l\}}
$$

and

$$
[\Theta_{\hat{\sigma}}, H_{\hat{\sigma}}] = -\sum_{\substack{j=2 \\ j\notin\sigma}}^{n} \left[\Theta_{\hat{\sigma}\hat{j}}\partial_j, 2\sum_{\substack{l=1 \\ l\neq j,\sigma}}^{n} u_{jl}\right] - \sum_{\substack{2\leq k<l\leq n \\ k,l\notin\sigma}} \left[\Theta_{\hat{\sigma}\hat{k}\hat{l}}, 2\left(u_{1k}+u_{1l}\right)\right] X_{\{k,l\}},
$$

and equality

$$
\left[\Theta_{\{1,k,l\}}, H_{\{1,k,l\}}\right] = -\left[\partial_1, 2\left(u_{1k}+u_{1l}\right)\right] X_{\{k,l\}} - \left[\Theta_{\{1,l\}}\partial_k, 2\left(u_{1k}+u_{kl}\right)\right] - \left[\Theta_{\{1,k\}}\partial_l, 2\left(u_{1l}+u_{kl}\right)\right], \tag{22}
$$

The last equality can be obtained by direct calculation or using formulae for the three-particle case. Once all these substitutions are made it can be easily seen that all terms in (17) are canceled. This completes the proof of Theorem 1.

7 Concluding remarks.

A more general class of the deformed CM operators related to any Lie superalgebra has been recently introduced in [16] by Sergeev and Veselov, who found also recurrent formulae for the quantum integrals in trigonometric case for the classical series. In the elliptic case the integrability of these systems is an open problem. Within this approach the operator (1) considered in our paper corresponds to the Lie superalgebra $sl(n-1,1)$. The author hopes that the technique developed in this paper can be applied to more general deformed elliptic CM operators related to Lie superalgebra $sl(n,p)$.

In this relation it is worth mentioning that as one can see from our formulae (6) the parameters $m = 1/l, l = 1, 2, 3\ldots$ play a special role in the theory of the deformed CM operators. It is interesting to compare this with the results of the papers [16, 17], where exactly these particular values of the parameter arise in relation with the strata in the discriminant variety consisting of polynomials having a root of multiplicity l.

8   Acknowledgement

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